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THE EFFECT OF THERMAL CONDUCTION OF THE WALL UPON CONVECTION FROM A SURFACE IN A LAMINAR BOUNDARY LAYER

ZEEV ROTEM†

Department of Mechanical Engineering, The University of British Columbia, Vancouver, B.C., Canada

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Abstract—This paper considers the influence of wall thermal conduction upon the interface temperature profile for the case of a thin heat dissipating wall cooled by forced laminar convection. A method for the rapid approximate calculation of both the temperature and the film coefficient is given, for the two cases of an almost isothermal wall and a wall of almost constant flux.

NOMENCLATURE

- Bi, Biot number, equation (5a);
- H, film convection coefficient;
- K, thermal conductivity;
- l, length of heated section of wall;
- L, reference length;
- q, quantity of heat;
- Pr, Prandtl number;
- Re, Reynolds number;
- T, temperature;
- U, reference velocity;
- x, coordinate;
- X, modified coordinate;
- α, constant;
- β, dimensionless shear stress at the wall;
- γ, defined in equation (15);
- δ, thickness of heated wall;
- Δ, l/L;
- ϵ, ε, parameters;
- ϑ, θ, dimensionless temperatures;
- Θ, modified dimensionless wall temperature;
- ρ, density of fluid;
- τ, shear stress.

Superscripts

- $*$, refers to start of heating section;
- $'$, signifies modified variable.

Subscripts

- o, refers to $x = 0$;
- 1, refers to dimensional variables;
- f, refers to fluid;
- s, refers to solid;
- w, refers to wall.

† Professor. This work was carried out while the author held a Research Fellowship at the Department of Chemical Engng., Stanford University, Stanford, Cal. U.S.A.

INTRODUCTION

In the analysis of problems of heat convection from or to solid boundaries, both in internal and external flows, it is common to specify as a boundary condition at the solid confining surface a given temperature or flux distribution.

In most cases arising in practice the temperature and flux distributions actually obtained on the boundary are dependent upon the physical properties of the confining wall itself, and are seldom known *a priori*. It is therefore only under idealized conditions that such commonly specified boundary conditions as an isothermal interface or a constant temperature gradient at the wall surface are ever achieved.

That the problem has nevertheless received scant attention is due to its complexity. Thus, for the comparatively simple case of the two-dimensional laminar flow of an incompressible, constant property Newtonian fluid over a constant property wall a complete treatment would involve the simultaneous solution of the boundary layer momentum and energy equations, as well as the diffusion equation with heat sources, for the wall. It is the purpose of the present paper to consider approximate solutions for this case, using the supplementary simplification that the wall is sufficiently thin in a direction normal to the interface, so that temperature variations in that direction may be neglected.

It is intuitively obvious that a non-zero wall conductivity will have the effect of smoothing-out the temperature profile, causing a flux in the wall, in a surface parallel to the main flow, and in a direction opposed to the temperature gradient. In the extreme case of an infinitely conductive wall a constant temperature is created at the interface. Conversely, if there is no thermal conductivity in the wall, the interface flux will be specified by the local source strength in the wall.

ANALYSIS

Consider the equation of unidimensional heat conduction for a confining wall of thickness δ and heated length l containing distributed heat sources of strength q_w''' per unit volume and per unit time. It is assumed that $\delta \ll l$. The configuration is illustrated in Fig. 1. Then,

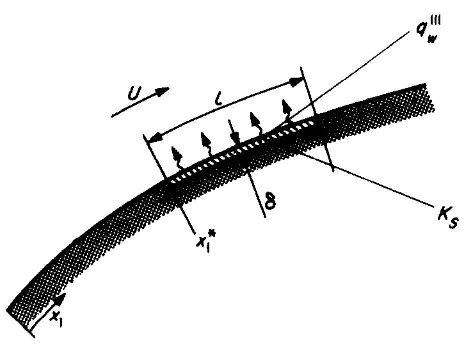

FIG. 1.

$$\left.\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x_1}\left[K_s(T, x_1)\frac{\mathrm{d}T}{\mathrm{d}x_1}\right] + q_w'''(T, x_1)\\ &\quad - \frac{H(x_1)}{\delta}[T(x_1) - T_\infty(x_1)] = 0\\ &\qquad\qquad (x_1^* < x_1 \leqslant x_1^* + l),\\ &q_w'''(T) \equiv 0 \qquad (x_1 \leqslant x_1^*, x_1 > x_1^* + l).\end{aligned}\right\} \quad (1)$$

Here x_1 is the (curvilinear) coordinate in the interface, x_1^* is the coordinate at the start of the heated surface, $T(x_1)$ is the wall temperature, $T_\infty(x_1)$ is the temperature of the fluid outside the wall boundary layer, $K_s(T)$ is the thermal conductivity of the wall material and $H(x_1)$ is the film convection coefficient. In the following the wall material will be assumed to possess constant physical properties over the length of the heated section. The boundary conditions at the ends of the heated segment of the wall will depend upon the physical properties of the wall material. We have,

$$\left.\begin{aligned}T(x_1 \to 0) &= T_\infty\\ T(x_1 \to \infty) &= T_{\infty 2}\end{aligned}\right\} \quad (2)$$

while at points $x_1 = x_1^*$ and $x_1 = x_1^* + l$ the following conditions obtain,

$$\left.\begin{aligned}T_x(x_1^* + 0) &= \frac{K_s(x_1^* - 0)}{K_s(x_1^* + 0)}\, T_x(x_1^* - 0)\\ T_x(x_1^* + l - 0) &= \frac{K_s(x_1^* + l + 0)}{K_s(x_1^* + l - 0)}\\ &\qquad \times T_x(x_1^* + l + 0).\end{aligned}\right\} \quad (3)$$

In particular if the wall material outside the heated segment is of zero thermal conductivity, then from (3),

$$T_x(x_1^* + 0) = 0 \qquad T_x(x_1^* + l - 0) = 0, \quad (4)$$

and equations (1), (4) will describe the behaviour of a heated strip. For convenience we shall now introduce dimensionless variables as follows:

$$x = x_1/L \qquad \Delta = l/L \quad \text{and}$$

$$\vartheta(x) = \frac{K_s(T - T_\infty)}{q_w''' L^2} \quad (5)$$

where L is some suitable reference length. The Biot modulus is defined as,

$$Bi = \frac{H \,.\, L^2}{K_s \,.\, \delta}. \tag{5a}$$

Introducing these variables into equation (1) gives (for the particular case of constant K_s, T_∞ and q_w''')

$$\frac{d^2\vartheta}{dx^2} - Bi\vartheta + 1 = 0. \tag{6}$$

THE CASE OF ALMOST CONSTANT WALL TEMPERATURE

It is now necessary to derive a suitable expression for the dependence of Bi upon the other variables of the problem. We shall be concerned in practice most often with cases of a sectionally almost isothermal wall, or else an almost constant flux wall. For the first case Lighthill [1] has developed a suitable expression for the film coefficient

$$H(x) = \frac{K_f}{L}\frac{Re^{\frac{1}{2}}}{\Gamma(\frac{4}{3})}\left(\frac{Pr}{9}\right)^{\frac{1}{3}} \sqrt{[\beta(x)]} \times \int_0^x \left\{ \int_{x^*}^x \sqrt{[\beta(\sigma)]}\, d\sigma\right\}^{-\frac{1}{3}} d\theta_w(x^*). \tag{7}$$

Here K_f is the thermal conductivity of the fluid; $\beta(x)$ is the dimensionless shear-stress at the wall,

$$\beta(x) = \frac{\tau_w}{\rho U^2} Re^{\frac{1}{2}},$$

where U is some reference velocity, θ_w is the dimensionless excess wall temperature, and the integral in equation (7) is to be interpreted in the Stieltjes sense. This expression for the film coefficient may now be introduced into (5a) and into (6). We shall consider in more detail the case of a conducting strip of the wall, embedded in non-conducting material (Fig. 1). In this particular case it is useful to modify the co-ordinates as follows:

$$\left.\begin{aligned} X &= x/\Delta \\ \Theta &= \frac{K_f}{K_s}\frac{L}{\delta}\frac{Re^{\frac{1}{2}}}{\Gamma(\frac{4}{3})}\left(\frac{Pr}{9}\right)^{\frac{1}{3}}\frac{\vartheta}{\Delta^{\frac{1}{3}}}. \end{aligned}\right\} \tag{8}$$

The parameter ε is defined by

$$\varepsilon = \frac{K_s}{K_f}\frac{\delta}{L}\frac{\Gamma(\frac{4}{3})}{Re^{\frac{1}{2}}}\left(\frac{9}{Pr}\right)^{\frac{1}{3}} \Delta^{-\frac{5}{3}}. \tag{9}$$

Thereupon equation (6) now reads,

$$\varepsilon\Theta_{XX} - \sqrt{[\beta(X)]}\int_0^X \frac{d\Theta(X^*)}{\left\{\int_{X^*}^X \sqrt{[\beta(\nu)]}\, d\nu\right\}^{\frac{1}{3}}} + 1 = 0 \qquad (\varepsilon \gg 1) \tag{10}$$

with boundary conditions,

$$\Theta_X(X^*) = \Theta_X(X^* + 1) = 0. \tag{11}$$

For the case of an almost isothermal wall, the parameter ε will have a very large numerical value. Hence it will be possible to solve equation (10) in a regular pertubation expansion in $1/\varepsilon$ for a variety of cases. As an example we solve here the simple case of constant $\beta(X)$, which is not directly applicable to the case of external "wedge flows", especially if the heated surface is long. It is, however, applicable to the case of hydrodynamically "fully developed" internal flows (with certain supplementary restrictions, to be discussed below). The expression for the temperature becomes,

$$\Theta(X) = \tfrac{2}{3}[\beta(X)]^{-\frac{1}{3}} + \frac{1}{\varepsilon}\left\{\tfrac{3}{5}\,.\,(X - X^*)^{\frac{5}{3}} - \tfrac{1}{2}(X - X^*)^2 - \frac{9}{140}\right\} + \frac{1}{\varepsilon^2}[\beta(X)]^{+\frac{1}{3}}\,.\left\{\frac{27}{40} \times \left[\frac{(X - X^*)^{\frac{10}{3}}}{7} - \frac{(X - X^*)^{\frac{11}{3}}}{11}\right] - \frac{81}{1400}(X - X^*)^{\frac{5}{3}} + \alpha\right\} + 0|\varepsilon^{\frac{1}{3}}|. \tag{12}$$

The constant α is determined from the condition that the highest order term retained in the expansion (12) will fulfill the equation (10) *in the mean*. In the present case

$$\alpha = \frac{3361}{254800} = 0{\cdot}01319.$$

Once the temperature profile in the wall has been determined, equation (7) will yield the film convection coefficient. It is seen that near the leading edge of the plate this coefficient will be markedly reduced by the effect of wall conduction.

The solution (12) will prove useful even in the case in which the shear stress at the wall depends upon X, provided the extent of the heated length of the wall is not too great. In this case it is sufficient to introduce an average value of $\beta(X)$, [2],

$$\bar{\beta}(X) = \left\{ \frac{\int_{X}^{X^*} \sqrt{[\beta(\zeta)]}\, d\zeta}{X - X^*} \right\}^2 \qquad (13)$$

into equation (12) and an approximate value for $\Theta(X)$ will then be obtained.

THE CASE OF SMALL WALL CONDUCTIVITY

It is apparent that for vanishingly small ε, a perturbation expansion based upon equation (10) would become singular. Moreover, in the limit of $\varepsilon \to 0$ a constant-flux interface would be obtained, so that it will be appropriate to replace equation (7) by an expression suitable for that case [1],

$$H(x) = \frac{K_f}{L}\, \Gamma(\tfrac{2}{3})\,.\,(3Pr)^{\frac{1}{3}}\,.\,Re^{\frac{1}{2}} \left\{ \int_{x^*}^{x} \frac{dx}{[\gamma(x) - \gamma(x^*)]^{\frac{1}{3}}} \right\}^{-1} \qquad (14)$$

with,

$$\gamma(x) = \int_0^x \sqrt{[\beta(\zeta)]}\, d\zeta. \qquad (15)$$

The counterparts to equations (8) through (10) become,

$$\Theta = \frac{K_f}{K_s}\,.\,\frac{L}{\delta}\,.\,\Gamma(\tfrac{2}{3})\,.\,(3Pr)^{\frac{1}{3}}\, Re^{\frac{1}{2}}\, \vartheta \qquad (16)$$

$$\epsilon = \frac{K_s}{K_f}\frac{\delta}{L}\, [\Gamma(\tfrac{2}{3})\, Re^{\frac{1}{2}}\, (3Pr)^{\frac{1}{3}}]^{-1} \qquad (17)$$

and,

$$\epsilon\Theta_{xx} - \frac{\Theta}{\int_{x^*}^{x} [\gamma(x) - \gamma(x^*)]^{-\frac{1}{3}}\, dx} + 1 = 0. \qquad (\epsilon \ll 1). \qquad (18)$$

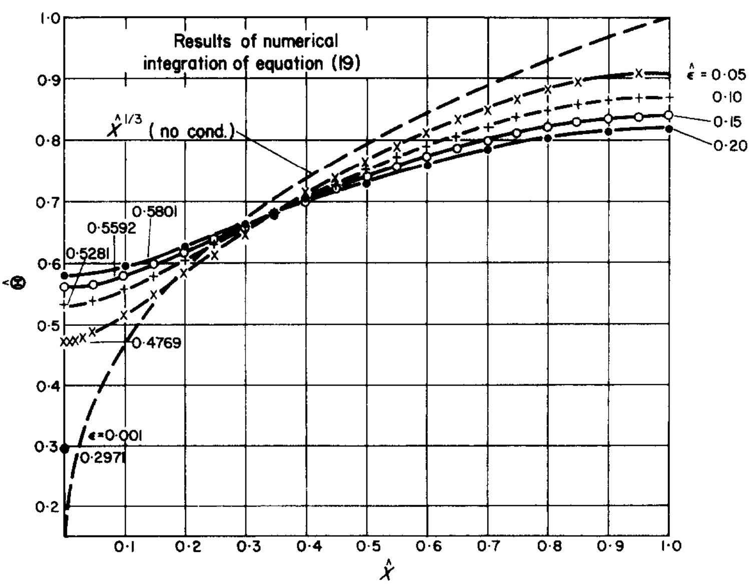

FIG. 2.

The boundary conditions for equation (18) remain equations (11).

As an example, we shall again solve the case of a constant $\beta(x)$. It is found convenient to replace

$$\hat{\Theta} = \tfrac{1}{3}(\beta/\Delta)^{\frac{1}{3}}\,\Theta, \qquad \hat{\epsilon} = \frac{3\epsilon}{(\beta\Delta^5)^{\frac{1}{3}}}$$

$$\text{and} \qquad \hat{X} = \frac{x - x^*}{\Delta}$$

and then to solve numerically the resultant equations,

$$\hat{\epsilon}\,\hat{\Theta}_{\hat{X}\hat{X}} - \frac{\hat{\Theta}}{\hat{X}^{\frac{1}{3}}} + 1 = 0 \qquad (\hat{\epsilon} \ll 1) \qquad (19)$$

$$\hat{\Theta}_{\hat{X}}\,(0) = \hat{\Theta}_{\hat{X}}\,(1) = 0. \qquad (20)$$

Some results, computed with the aid of an Adams–Bashforth predictor corrector routine are shown in the graph, Fig. 2. There the variation of $\hat{\Theta}$ with $\hat{X}$ is given for several values of the parameter $\hat{\epsilon}$. The intercepts of these curves on the $\hat{X} = 0$ axis are also indicated numerically.

As already pointed out above, a perturbation expansion based upon equation (19) would become singular. Therefore the influence of the small but non-zero conduction effect is felt mainly in a narrow "boundary layer" near the leading edge of the heated strip. There the first and second terms of equation (19) must become of comparable order of magnitude even in the limit of $\hat{\epsilon} \to 0$. We are therefore immediately able to predict that near the leading edge,

$$\hat{\Theta} \propto \hat{\epsilon}^{\frac{1}{4}} \qquad (\hat{\epsilon},\, \hat{X} \text{ small}). \qquad (21)$$

This prediction is compared in Table 1 with data computed from equation (19). The agreement is seen to be satisfactory.

Table 1

$\hat{\epsilon}$	0·05	0·10	0·15	0·20
$(\hat{\epsilon}/0{\cdot}10)^{\frac{1}{4}}$	0·8707	1·0000	1·0844	1·1488
$\dfrac{\hat{\Theta}_0(\hat{\epsilon})}{\hat{\Theta}_0(\hat{\epsilon} = 0{\cdot}10)}$	0·9029	1·0000	1·0589	1·0987

DISCUSSION

Comparatively little has been reported in the relevant literature on the effect of wall thermal conduction upon interface temperature or flux distribution. Some recent work dealing with the subject is given in references [3] and [4]. In the present paper a simple approximate method for the evaluation of the wall conduction effect is described, and a particular example dealing with distributed heat sources of constant intensity is worked out in detail. The formulae given are suitable mainly for laminar boundary layers in external flows. They may, however, be extended to the case of turbulent flows with a laminar sublayer as well. In internal flows their usefulness is limited to the vicinity of the start of the heated part of the wall, as no account is taken in the present method of the variation in bulk temperature with x.

The results of the case of a wall approaching the condition of constant flux at the interface are summarized in the graph Fig. 2. It will be noted that near the leading edge of the heated surface the effect of even a very small wall conduction is notable. For an actual practical case the following values were computed:

> Strip made of Constantan, heated electrically, embedded in insulating material and cooled by forced convection of oil over its free surface. $\delta/L_{\text{ref}} = 0{\cdot}0368$, $R_{eL} = 91{\cdot}5$, ratio of the coefficients of thermal conductivity 103·2, $\Delta = 0{\cdot}59$. The value of $\hat{\epsilon}$ is 0·13.

Equations (7) and (14), which were used in this work to evaluate the film coefficients, are essentially asymptotic solutions of the thermal boundary-layer equations for large values of the Prandtl number. It has been shown [1] that their usefulness extends down to Pr of about 0·7.

Analogous expressions for the case of very *low* Prandtl number are also available. However, fluids of small Pr will in general possess a very high thermal conductivity so that the influence of longitudinal temperature gradients in the *fluid* (neglected in these derivations) may no longer be ignored. Therefore the extension of the method described here to those fluids is not warranted.

The effects of the sharp discontinuity in wall temperature at the leading edge of the heated section of the wall are not limited to the wall itself. A more thorough investigation would have to take into account the axial conductivity in the fluid over the leading edge as well [5] even for the case of values of *Pr* which are not small. No account is taken of that effect in the work reported here.

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Résumé—On considère l'influence de la conduction thermique dans une paroi sur le profil de température à l'interface pour le cas d'une paroi mince avec dissipation de chaleur interne refroidie par convection forcée laminaire. On donne une méthode pour calculer rapidement d'une façon approchée la température et le coefficient de film, dans les deux cas d'une paroi presque isotherme et d'une paroi avec un flux de chaleur presque uniforme.

Zusammenfassung—Es wird hier der Einfluss der Wärmeleitfähigkeit der Wand auf das Temperaturprofil der angrenzenden Schicht behandelt für den Fall einer dünnen Wand mit Wärmedissipation wobei die Kühlung durch erzwungene laminare Konvektion erfolgt. Eine Methode zur schnellen, näherungsweisen Berechnung sowohl des Temperatur- als auch des Filmkoeffizienten ist angegeben für die beiden Fälle der nahezu isothermen Wand und des nahezu konstanten Wärmeflusses.

Аннотация—В статье рассматривается влияние теплопроводности стенки на распределение границы раздела для случая тонкой рассеивающей тепло стенки при вынужденной ламинарной конвекции. Излагается метод быстрого приближенного расчета как температуры, так и пленочного коэффициента теплообмена для двух случаев: почти изотермической стенки и стенки с почти постоянным тепловым потоком.